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High-Order Realisations of the Para-Fermi Algebra with Parafield Operators

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Abstract

Using second-order realisations of Lie algebras by means of creation and annihilation parafield operators the generators of the para-Fermi algebras are expressed as high-order polynomials of para-Bose or para-Fermi creation and annihilation operators.

1. *Introduction*

The possibility of realising any Lie algebra by means of bilinear combinations of creation and annihilation parafield (Green) operators (Kademova, 1970a) enabled us to develop a method in a series of papers (Kademova, 1970b; Kademova & Kálnay, 1970; Kademova & Kraev, 1969) in which the generators of the para-Fermi algebra are expressed as functions of bilinear combinations of the creation and annihilation para-Bose or para-Fermi operators. The realisations of Lie algebras as second-order polynomials of parafield operators (Kademova & Palev, 1970a) opens a way for new realisations of the para-Fermi algebra generators as higher-order polynomials of the parafield operators.

In Section 2 isomorphic mappings of the para-Fermi algebra into the second-order polynomials of the Green operators are defined using the embedding of an arbitrary matrix realisation of the para-Fermi algebra into the Lie algebra $\mathscr{A}_\epsilon$ of corresponding dimensionality [$\mathscr{A}_+ = sp(2k)$, $\mathscr{A}_- = o(k,k)$]. With the help of these second-order realisations one can easily find higher-order realisations of the para-Fermi algebra (Section 3).

2. *Embedding of the Para-Fermi Algebra into Second-Order Polynomials of Parafield Operators*

We consider the algebra† $\mathscr{U}(n,\epsilon)$ generated by the entities a_i, $\overset{+}{a}_j$, i, $j = 1, \ldots, n$ in which the following relations hold

$$\begin{aligned} [\tfrac{1}{2}[\overset{+}{a}_i, a_j]_\epsilon, \overset{+}{a}_k]_- &= \delta_{jk}\overset{+}{a}_i \\ [\tfrac{1}{2}[a_i, a_j]_\epsilon, a_k]_- &= 0 \end{aligned} \tag{2.1}$$

where $\epsilon = \pm$.

† A precise mathematical definition is given in Kademova (1970a).

By use of the Green Ansatz†

$$a_i = \sum_{\alpha=1}^{p} a_i^{\alpha}$$
$$\overset{+}{a}_i = \sum_{\alpha=1}^{p} \overset{+}{a}_i^{\alpha} \tag{2.2}$$

we can embed $\mathscr{U}(n, \epsilon)$ into the algebra‡ $\mathscr{U}(n,p,\epsilon)$ generated by 2^{np} elements a_i^{α}, $\overset{+}{a}_j^{\beta}$, $i, j = 1, \ldots, n$, α, $\beta = 1, \ldots, p$, defined by the relations

$$[a_i^{\alpha}, \overset{+}{a}_j^{\alpha}]_{-\epsilon} = \delta_{ij}, [a_i^{\alpha}, a_j^{\alpha}]_{-\epsilon} = [\overset{+}{a}_i^{\alpha}, \overset{+}{a}_j^{\alpha}]_{-\epsilon} = 0$$
$$[a_i^{\alpha}, \overset{+}{a}_j^{\beta}]_{\epsilon} = [a_i^{\alpha}, a_j^{\beta}]_{\epsilon} = [\overset{+}{a}_i^{\alpha}, \overset{+}{a}_j^{\beta}]_{\epsilon} = 0 \quad \text{if } \alpha \neq \beta \tag{2.3}$$

In what follows we shall find some new realisations of the algebra $\mathscr{U}(n,-)$ by means of the generators of the algebra $\mathscr{U}(2^{np}, \epsilon)$.

It has been pointed out by Green that for a fixed p (parastatistics of the parafield operators a_i, $\overset{+}{a}_j$) a matrix realisation of dimensionality 2^{np} for the para-Fermi algebra $\mathscr{U}(n,-)$ exists.

We embed the generators $F^p{}_i$, $\overset{+}{F}{}^p{}_j$, $i, j = 1, \ldots, n$, of this matrix realisation, for which the relations (2.1) hold, into $2^{np+I} \otimes 2^{np+I}$ matrices

$$\mathbf{F}^p{}_i = \begin{pmatrix} F^p{}_i & 0 \\ 0 & -(F^p{}_i)^T \end{pmatrix}$$
$$\overset{+}{\mathbf{F}}{}^p{}_i = (\mathbf{F}^p{}_i)^+ \tag{2.4}$$

which, as it is easily seen, satisfy the same relations (2.1).

The matrices (2.4) belong to the algebra $\mathscr{A}_{\epsilon}$ $[\mathscr{A}_+ = sp(2^{np+I}), \mathscr{A}_- = o(2^{np}, 2^{np})]$.

We denote by G_{ϵ} the group which preserves the bilinear form with a matrix

$$\beta = \begin{pmatrix} 0 & I \\ -\epsilon I & 0 \end{pmatrix}$$

$(G_+ = Sp(2^{np+I}), G_- = o(2^{np}, 2^{np})$. Since the group G_{ϵ} is a semisimple one it is isomorphic to the adjoint group of the algebra $\mathscr{A}_{\epsilon}$, and therefore the mapping θ

$$\mathbf{F}^p{}_i \to \tilde{\mathbf{F}}^p{}_i = g_{\epsilon} \mathbf{F}^p{}_i g_{\epsilon}^{-1}$$
$$\overset{+}{\mathbf{F}}{}^p{}_i \to \overset{+}{\tilde{\mathbf{F}}}{}^p{}_i = g_{\epsilon} \overset{+}{\mathbf{F}}{}^p{}_i g_{\epsilon}^{-1} \tag{2.5}$$

where $g_{\epsilon} \in G_{\epsilon}$ is an automorphism of the algebra $\mathscr{A}_{\epsilon}$.

Finally, we define a set of new entities by the mapping $\theta^p_{q\epsilon}$

$$\tilde{\mathbf{F}}^p{}_i \to \mathscr{F}^p_{q\epsilon i} = \tfrac{1}{2} \tilde{\varphi}_{q\epsilon} g_{\epsilon} \mathbf{F}^p{}_i g_{\epsilon}^{-1} \varphi_{q\epsilon}$$
$$\overset{+}{\tilde{\mathbf{F}}}{}^p{}_i \to \overset{+}{\mathscr{F}}{}^p_{q\epsilon i} = \tfrac{1}{2} \tilde{\varphi}_{q\epsilon} g_{\epsilon} \overset{+}{\mathbf{F}}{}^p{}_i g_{\epsilon}^{-1} \varphi_{q\epsilon} \tag{2.6}$$

† For more details see Green (1953), where this has been introduced for the first time.
‡ See Kademova & Palev (1970b).

where

$$\tilde{\varphi}_{q\epsilon} = (\overset{+}{A}, A), \qquad \varphi_{q\epsilon} = \begin{pmatrix} A \\ + \\ -\epsilon A \end{pmatrix}, \qquad A = (a_1, \ldots, a_{2^{np}})$$

$\overset{+}{A} = (\overset{+}{a}_1, \ldots, \overset{+}{a}_{2^{np}})$, $\epsilon\overset{+}{A} = (\epsilon\overset{+}{a}_1, \ldots, \epsilon\overset{+}{a}_{2^{np}})$, where a_i, $\overset{+}{a}_j$ are para-Bose operators of parastatistics q for positive ϵ, and para-Fermi ones of parastatistics q for ϵ negative.

Now we can prove the following theorem:

Theorem

The entities $\mathscr{F}^p_{q\epsilon i}$, $\overset{+}{\mathscr{F}}{}^p_{q\epsilon j}$, $i, j = 1, \ldots, n$, defined through the mapping $\theta^p_{q\epsilon}$, generate a para-Fermi algebra.

Proof: Since the mapping $\theta^p_{q\epsilon}$ is one-to-one (see Kademova & Palev, 1970a), it is enough to prove that the Green commutation relations (2.1) are preserved. One can easily check using the results of the same paper that:

$$\begin{aligned}
[\tfrac{1}{2}[\overset{+}{\mathscr{F}}{}^p_{q\epsilon i}, \mathscr{F}^p_{q\epsilon j}]_-, \overset{+}{\mathscr{F}}{}^p_{q\epsilon k}]_- &= [\tfrac{1}{2}\tilde{\varphi}_{q\epsilon}\, g_\epsilon \tfrac{1}{2}[\overset{+}{\mathbf{F}}{}^p{}_i, \mathbf{F}^p{}_j]_- g_\epsilon^{-1}\, \varphi_{q\epsilon}, \overset{+}{\mathscr{F}}{}^p_{q\epsilon k}]_- \\
&= \tfrac{1}{2}\tilde{\varphi}_{q\epsilon}\, g_\epsilon [\tfrac{1}{2}[\overset{+}{\mathbf{F}}{}^p{}_i, \mathbf{F}^p{}_j]_-, \overset{+}{\mathbf{F}}{}^p{}_k]_-\, g_\epsilon^{-1} \varphi_{q\epsilon} \\
&= \delta_{jk}\, \tfrac{1}{2}\tilde{\varphi}_{q\epsilon}\, g_\epsilon\, \overset{+}{\mathbf{F}}{}^p{}_i\, g_\epsilon^{-1}\, \varphi_{q\epsilon} = \delta_{jk}\, \overset{+}{\mathscr{F}}{}^p_{q\epsilon i}
\end{aligned}$$

and also

$$[\tfrac{1}{2}[\mathscr{F}^p_{q\epsilon i}, \mathscr{F}^p_{q\epsilon j}]_-, \mathscr{F}^p_{q\epsilon k}]_- = 0$$

All the other relations, which can be obtained from here, using the formal conjugation rules and Jacobi identity, are also satisfied.

So we have proved that the mapping $\theta^p_{q\epsilon}$ of the para-Fermi algebra, generated by the elements $F^p{}_i$, $\overset{+}{F}{}^p{}_j$, $i, j = 1, .., n$, into the second-order polynomials $\mathscr{F}^p_{q\epsilon i}$, $\overset{+}{\mathscr{F}}{}^p_{q\epsilon j}$ of the Green operators of parastatistics q, is a Green isomorphism. In this way we have given a second-order realisation of n para-Fermi operators of parastatistics p by means of 2^{np} para-Bose or para-Fermi operators ($\epsilon = \pm 1$) of arbitrary order of parastatistics q.

3. *Higher-Order Realisations of the Para-Fermi Algebra*

Here we briefly sketch the idea of how one can get higher-order realisations of the algebra $\mathscr{U}(n,-)$, using the second order realisations obtained in the previous section. In a similar way as before we can define the mapping $\theta^p_{qq'-}$

$$\begin{aligned}
\tilde{\mathbf{F}}^p{}_i &\to \mathscr{F}^p_{qq'-i} = \tfrac{1}{2}\tilde{\varphi}_{qq'-}\, g_-\, \mathbf{F}^p{}_i\, g_-^{-1}\, \varphi_{qq'-} \\
\overset{+}{\tilde{\mathbf{F}}}{}^p{}_i &\to \overset{+}{\mathscr{F}}{}^p_{qq'-i} = \tfrac{1}{2}\tilde{\varphi}_{qq'-}\, g_-\, \overset{+}{\mathbf{F}}{}^p{}_i\, g_-^{-1}\, \varphi_{qq'-}
\end{aligned} \tag{3.1}$$

where

$$\tilde{\varphi}_{qq'-} = (\overset{+}{A'}, A'), \qquad \varphi_{qq'-} = \begin{pmatrix} A' \\ + \\ A' \end{pmatrix}, \qquad A' = (\mathscr{F}^{q}_{q'\epsilon 1}, \ldots, \mathscr{F}^{q}_{q'\epsilon 2^{np}})$$

$\overset{+}{A'} = (\overset{+}{\mathscr{F}}{}^{q}_{q'\epsilon 1}, \ldots, \overset{+}{\mathscr{F}}{}^{q}_{q'\epsilon 2^{np}})$, $\mathscr{F}^{q}_{q'\epsilon i}$, $\overset{+}{\mathscr{F}}{}^{q}_{q'\epsilon j}$, $i, j = 1, \ldots, 2^{np}$, are defined by the formula (2.6) as second-order polynomials of $2^{2^{np} \cdot q}$ para-Bose or para-Fermi operators, and therefore $\mathscr{F}^{p}_{qq'-i}$, $\overset{+}{\mathscr{F}}{}^{p}_{qq'-j}$ are fourth-order polynomials of these operators.

Following this procedure one can realise the para-Fermi algebra generators as higher-order polynomials of parafield operators by increasing the number of the generators of the para-Bose or para-Fermi algebra by means of which the realisation is constructed.

4. *Discussion*

One can consider the transformations induced in the Fock space of para-Bose or para-Fermi operators by the para-Fermi algebra generators (2.6) or (3.1) in a similar way as in Kademova (1970b), Kademova & Kálnay (1970) and Kademova & Kraev (1970).

The fact that the algebra $\mathscr{U}(n, \epsilon)$ can be embedded into $\mathscr{U}(n, p, \epsilon)$ allows us to extend the space of the representations to the Fock space of the quasifield operators spanned on the vectors

$$\prod_{i,\alpha} (\overset{+}{a}{}_i^{\alpha})^{m_{i\alpha}} |0\rangle$$

and to consider the induced transformations there.

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